

**Exercise 2.13.** Let  $A$  be a non-empty subset of  $X$ . A point  $a \in X$  is called a boundary point of  $A$  if  $B(a, r) \cap A \neq \emptyset$  and  $B(a, r) \cap A^c \neq \emptyset$  for all  $r > 0$ , where  $A^c$  denotes the complement of  $A$  in  $X$ . The set of all boundary points, write  $\partial A$ , of  $A$  is called the boundary of  $A$ .

- (i) Find the boundaries of  $\mathbb{Z}$  and  $\mathbb{Q}$  in  $\mathbb{R}$ .
- (ii) Let  $X = (0, 1) \cup (2, 3)$ . Find the boundary of the set  $(0, 1)$  in  $X$ .
- (iii) Show that the boundary  $\partial A$  is a closed subset of  $X$ .
- (iv) Show that  $\bar{A} = A \cup \partial A$ .

**Solution:**

(i) For any integer  $n \in \mathbb{Z}$ , we consider arbitrary neighborhood around  $n$ ,  $(n - \delta, n + \delta)$  for any  $\delta > 0$ .

Note that  $n \in (n - \delta, n + \delta)$ , then  $B(n, \delta) \cap \mathbb{Z} \neq \emptyset$ .

Also  $n + \min\{\delta, \frac{1}{2}\} \in B(n, \delta) \cap \mathbb{Z}^c$ . Thus  $\mathbb{Z} \subseteq \partial \mathbb{Z}$ .

On the other hand, let  $x \in \mathbb{Z}^c$  and  $r_0 = \min\{|x - n|, n \in \mathbb{Z}\} > 0$ .

Then  $B(x, r_0) \cap \mathbb{Z} = \emptyset$  and  $x \notin \partial \mathbb{Z}$ . Therefore  $\partial \mathbb{Z} \subseteq \mathbb{Z}$  and  $\partial \mathbb{Z} = \mathbb{Z}$ .

To find  $\partial \mathbb{Q}$ , we first recall following definition and facts.

**Def:** A set  $E$  of real numbers is said to be dense in  $\mathbb{R}$  provided between any two real numbers, there lies a member of  $E$ .

**Thm1:** The rational numbers are dense in  $\mathbb{R}$ .

**Thm2:** The irrational numbers are dense in  $\mathbb{R}$ .

Let  $a$  be any real number, and  $r$  be any real number. Then

$(a - r, a + r) \cap \mathbb{Q} \neq \emptyset$  (by Thm1).

$(a - r, a + r) \cap \mathbb{Q}^c \neq \emptyset$  (by Thm2).

Therefore  $\partial \mathbb{Q} = \mathbb{R}$ .

(ii) Let  $A = (0, 1)$ . Then  $A^c = (2, 3)$  in  $X$ .

For any  $a \in (0, 1)$  and  $r < 2 - a$ ,  $B(a, r) \cap A \neq \emptyset$ , but

$$B(a, r) \cap A^c = \emptyset.$$

Similarly, for any  $b \in (2, 3)$  and  $r < b - 1$ , we have

$$B(b, r) \cap A = \emptyset.$$

Hence  $\partial A = \emptyset$  in  $X$ .

(iii)

By the Definition 2.8.  $\partial A$  is closed in  $X$  if  $\overline{\partial A} = \partial A$ .

Using Definition 2.4,  $\overline{\partial A} = \partial A \cup D(\partial A)$  where  $D(\partial A)$

denote the set of all limit points in of  $\partial A$ . i.e.

$$D(\partial A) = \{x \in X : B(x, r) \setminus \{x\} \cap \partial A \neq \emptyset \text{ for all } r > 0\}.$$

Thus it suffices to show that  $D(\partial A) \subseteq \partial A$ . Let  $x \in D(\partial A)$

Then  $B(x, r) \cap \partial A \neq \emptyset$  for all  $r > 0$ .

For any  $r > 0$ , there exists  $x_r \in B(x, \frac{r}{2}) \cap \partial A$ , which satisfies

$$B(x_r, s) \cap A \neq \emptyset, \quad B(x_r, s) \cap A^c \neq \emptyset \text{ for all } s > 0.$$

Since  $B(x_r, \frac{r}{2}) \subseteq B(x, r)$ , then  $B(x, r) \cap A \neq \emptyset$ ,  $B(x, r) \cap A^c \neq \emptyset$ ,

for all  $r > 0$ . Thus  $x \in \partial A$  and  $\partial A$  is closed.

(iv) For  $x \in \partial A \setminus A$ , we have  $(B(x, r) \setminus \{x\}) \cap A \neq \emptyset$ ,  $\forall r > 0$ ,

then  $x \in D(A)$  and hence  $\partial A \setminus A \subseteq D(A)$ . It follows that

$$A \cup \partial A \subseteq A \cup D(A) = \overline{A}$$

On the other hand, if  $x \in D(A) \setminus A$ , we have

$(B(x, r) \setminus \{x\}) \cap A \neq \emptyset$  and  $B(x, r) \cap A^c \neq \emptyset$  for all  $r > 0$ .

which implies  $D(A) \setminus A \subseteq \partial A$ . Hence  $\overline{A} = A \cup D(A) \subseteq A \cup \partial A$ .

**Exercise 2.17.** (i) Let  $V$  be a subset of  $X$ . A point  $z \in V$  is said to be an interior point of  $V$  if there is  $r > 0$  such that  $B(z, r) \subseteq V$ . If we put  $\text{int}(V)$  the set of all interior points of  $V$ , show that  $\text{int}(V)$  is an open subset of  $X$ .

(ii) A metric  $d$  on  $X$  is said to be non-archimedean if it satisfies the strong triangle inequality, that is,  $d(x, y) \leq \max(d(x, z), d(z, y))$  for all  $x, y$  and  $z \in X$  (see also Example 1.2 (iv)). Show that if  $d$  is a non-archimedean metric on  $X$ , then for every closed ball  $\overline{B}(a, r) := \{x \in X : d(a, x) \leq r\}$  is an open set in  $X$ .

Solution:

(i) Let  $x \in \text{int}(U)$ . There exists  $r > 0$  such that  $B(x, r) \subseteq U$ . In particular, for any  $y \in B(x, \frac{r}{2})$

we have  $B(y, \frac{r}{2}) \subseteq B(x, r) \subseteq U$

so that  $y \in \text{int}(U)$  and

$$B(x, \frac{r}{2}) \subseteq \text{int}(U)$$

As  $x \in \text{int}(U)$  is arbitrary, we prove that  $\text{int}(U)$  is open.

(ii)

Let  $B = \overline{B}(a, r)$  be an arbitrary closed ball for some  $r > 0$ . Let  $x \in B$ . We will show that  $B(x, r) \subseteq B$ .

If  $y \in B(x, r)$ , then

$$d(a, y) \leq \max(d(a, x), d(x, y)) \leq \max(r, r) = r$$

Hence  $y \in \overline{B}(a, r)$ . Therefore  $B(x, r) \subseteq B$  for any  $x \in B$ , and  $B$  is open.

